LOBBY SEVEN

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"They write politics, we write government"

THE POINCARÉ CONJECTURE

A Tale of Rubber Bands and the Shape of the Universe

"Marty, it's perfect! You're just not thinking fourth-dimensionally!" – Emmett Brown

What is the shape of the universe? Are we even allowed to ask?

We know a lot about our universe. Painstaking work has taught us much about what's in it, how it began, how it is changing and even how it might end. A century of study of our existence on the vastest scales – Relativity – and the smallest scales – Quantum Mechanics – have given us a glimpse of what's out there. But we have more questions today than ever before.

But our story doesn't begin in the present. It begins in France at the end of the 19th century. It begins with Henri Poincaré, often considered the last universal mathematician. It begins with a piece of pure mathematics from the obscure discipline of topology.

- What is topology?
- What is the Poincaré Conjecture?
- What is the shape of the universe?

What is topology?

The last 150 years could be broadly described as an attempt to make mathematics more general – or to use the mathematical term, abstract. Number theory is the abstract version of arithmetic. Analysis is the abstract version of calculus. Abstract algebra is the abstract version of…well…algebra. Topology is the abstract version of geometry.

Topology is geometry on steroids.[1] You remember geometry – 9th grade, compass, straightedge, step-by-step proofs. In geometry, you were concerned with sizes, shapes and distances. A circle was the collection of all points equidistant from its center. An equilateral triangle had three sides of equal length.

Unhelpfully, Wikipedia defines topology as "the study of…topological spaces." More helpful is to think of topology as geometry, but where you are allowed to bend or stretch objects as long as you never tear or glue. It's like rubber or Play-Doh. In topology, a square can be smoothed (carefully) to make a circle. They are therefore the same thing in topology.[2] But they are both different from a figure "8," which you can only get to by pinching the center and gluing. Similarly, a donut and coffee mug are equivalent to each other, but both are different from a sphere or a cube.

[1] Or, maybe, something hallucinogenic.

[2] Technically, you would say they are "homeomorphic" but we'll use that word as little as possible because it's scary and this isn't a math class.

Another good example to describe topological equivalence is the alphabet. In the below table, each "class" of letters is equivalent to all of its class's members (and not equivalent to members of any other class). Each class can be described by the number of "holes" and "tails" of its members. Of course, the classes could be a bit different depending on your font.[3]

Class	*Members*	*Holes*	*Tails*
1	A, R	1	2
2	B	2	0
3	C, G, I, J, L, M, N, S, U, V, W, Z	0	0
4	D, O	1	0
5	E, F, T, Y	0	3
6	H, K	0	4
7	P, Q	1	1
8	X	0	4

Shapes that have different numbers of holes or tails can never be equivalent. Shapes with the same number of holes and tails can be different, "K" and "X". In the latter, the four tails meet at a central point, where in the former there is a small connector. Equivalence is transitive. In other words, if A is equivalent to B, and B is equivalent to C, then we know that A is equivalent to C. I'll leave the proof of this to the reader.

Topologists love to think and talk about properties of objects. In topology, we only care about those properties that are preserved among all objects that are equivalent. We saw some examples of this above – number of holes and tails are examples of topological properties. However, "has corners" is not a valid property in topology, because a square meets the criteria and a circle does not. Because it is not conserved between these equivalent shapes it is not a property in topology.[4]

We are going to work with some of these properties, so it makes sense to define them and give some examples. Note: these definitions are all non-technical – feel free to look them up on your own if you want the real meaning.[5]

Manifold: If an object "looks flat" when you zoom in on any small portion, it is a manifold. The surface of the Earth is a 2-dimensional manifold; when you stand on the street it appears flat, the curvature is too far away to see. The letter "X" is not a manifold; if you are at the intersection, there is no way to make it look flat (without tearing or gluing, which are prohibited in topology). A manifold with n dimensions is called a *n-Manifold*.

Boundary: The points you can get to both from inside and outside of an object are its boundary. In the below example, if the light blue area is the object, the dark blue is the boundary.

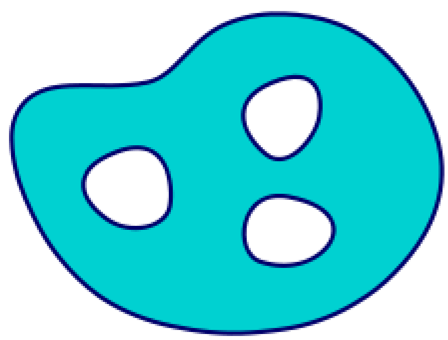


FIGURE 1

Not all objects have boundaries. For example, take a 1-dimensional circle; no matter how many times you go around, you will never reach an edge. The boundary can be itself part of the object, but it doesn't have to be.

[3] From Wikipedia.
[4] Sorry to keep going back, but Wikipedia has a nice list of topological properties. Here are two textbook excerpts of topological properties in action, if you want to dig deeper.

[5] As an example, I'm using the word "object" a lot; I don't think this word has a meaning in topology, but "subset of a topological space" is not exactly pithy language.

Compact: An object is compact if it includes its boundary and is of finite size. Figure 1 is compact if the object is considered to be the light- and dark-blue portions together. A donut is compact. A line, extending forever in both directions, is not of finite size and therefore not compact.

Closed: If an object is compact and has no boundary, it is closed. The skin of a balloon – which is 2-dimensional – is closed. Figure 1 is not closed; it is compact, but it has a boundary.

Simply Connected: This one is a bit trickier. Pretend you have a rubber band. A huge rubber band, very strong. No matter where on the Earth's surface you put the rubber band, it will constrict down to a single point and "snap-off" (if we ignore buildings and mountains, which topology allows). Because this is true for anywhere we put the rubber band, we say the Earth is simply connected.

FIGURE 2

On the other hand, if we put our rubber band around the handle of a coffee mug, it has no way to constrict completely. The same is true for a donut because the rubber band can go through the hole.

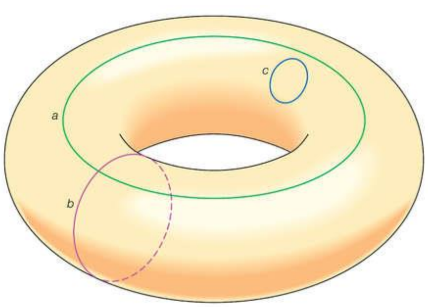


FIGURE 3

If there is any rubber band that doesn't "snap off" – like in the donut or coffee mug – then your object is not simply connected.

What is the Poincaré Conjecture?

When Fermat's Last Theorem was proven by Andrew Wiles in 1994, it was heavily reported and Wiles became a minor celebrity. When Grigori Perelman proved the Poincaré Conjecture in 2002 (verified in 2006) the popular response was more muted. However, virtually all mathematicians agree that the Poincaré Conjecture is the more "important" piece of mathematics.[6] Why was this?

There are a few reasons. First, Fermat's Theorem is much older – more than 350 years passed from Fermat's proposal until Wiles' proof, compared to exactly 100 from Poincaré to Perelman. Second, while Wiles is in not an attention hog, Perelman worked in near obscurity at Steklov Institute in Saint Petersburg (no offense intended, if there are any Steklovians reading this).

But more than anything else, the hesitancy to report on Perelman's breakthrough is probably because it is a bit more difficult to explain or visualize. It involves a 3-dimensional object that exists in 4-dimensional space. You have never existed in 4 dimensions of visible space and have no way to picture it.

[6] Of course, the unproven Riemann Hypothesis is an order of magnitude more important than either. But that's a story for another Volume…

An Aside: The strange existence of Grigori Perelman

The start of Grigori "Grisha" Perelman's life was typical enough; born in Leningrad to a professor of mathematics. He excelled in mathematics at a young age, taking advantage of late-Soviet programs to develop top talent of all types. The fall of Communism was timed precipitously for Perelman; he was able to take advantage of the post-doctoral circuit to study in the US and other places. His 1994 proof of the Soul Conjecture made him one of the world's top topologists. But he turned down offers from top global universities to return home. At the same time, global efforts towards solving Poincaré was advancing quickly. As a top talent in the topology, that Perelman might achieve the big prize was a possibility.

Rather than appearing in a journal, over a tantalizing period of eight months, on the public repository "arXiv", the math world watched Perelman's steady assault. The result was elegant; the final piece a svelte seven pages. The plaudits begin to roll in.

He turned down the Fields Medal – like a Nobel Prize for mathematics, but given out only quadrennially – saying "I don't want to be on display like an animal in the zoo." The Poincaré Conjecture was one of seven "Millenium Problems" chosen by the Clay Mathematics Institute for their difficulty and importance. Perelman, presumably not a rich man, turned down the $1 million prize, saying his contributions were no greater than those of Richard Hamilton. But Perelman didn't take the final step in Hamilton's 100-yard dash. He ran the last fifteen miles of his marathon.

There was some controversy proper credit for the proof, which clearly stressed Perelman. This may be the reason, he left the Steklov Institute in 2005. He hasn't published since; many say he has dropped out of mathematics.

There are only rumors of what he has been up to since. In preparing this article, the best I can say is that his current whereabouts are unknown.

Consider two disks – circles on a piece of paper with the interior filled in. Going to our definitions above, these disks are compact but not closed – they have boundaries. Say that the boundary of each disk is the Earth's equator. They are identically labelled, starting off the coast of Nigeria, passing east through Africa, the Indian Ocean, Indonesia, and so forth back where we started. Now, sew together the two disks so they align – Indonesia to Indonesia, Peru to Peru and so forth. Pull your construction apart at the poles; the result will be hollow. You now have the 2-dimensional surface of the Earth. In topology, we call this a 2-sphere. You can see easily that Earth's surface is compact, closed and simply connected. We therefore know that all 2-spheres must have these properties.

Now, consider 2 globes – different from above, because they aren't hollow any more. Just like the discs above, they are compact but not closed. With me so far? Because it is about to get tricky. Stitch these globes together, Brooklyn to Brooklyn, Iceland to Iceland, Sydney to Sydney. This is a 3-sphere. I know you can't picture it – because the 3-dimensional object you've created can only exist in 4-dimensional space.[7] I can't picture it either, nobody really can. But I can help you think about what it's like to live on this world.

Start in Detroit, on Globe A. Travel towards its core – remember these spheres were solid so you can do this. You pass through the core, then out to the other side at the surface of the Indian Ocean, Globe A.

FIGURE 4

Instantaneously, you pass to Indian Ocean, Globe B; they are stitched together. You keep going, through the core of Globe B, and back to the surface, Detroit on Globe B. But this is stitched to Detroit on Globe A; you are right back where you started.

[7] The 4th dimension isn't necessarily Einstein's famous "time." It isn't necessarily anything. Topology is so general that we can talk about a number of dimensions without caring what the dimensions are.

This is no different from what happens on our current planet. If you start at any point and travel in any direction, you'll get right back to where you started. Just because you can't picture it in higher dimensions doesn't mean it is so different. Without further ado:

> ***The Poincaré Conjecture:***
>
> **Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.**

A bit scary, I'm here to help. Remember that homeomorphic is just topology-speak for "the same as." An object is homeomorphic to the 3-sphere if you can change it into the 3-sphere by bending and stretching, without tearing or gluing. Simply connected, closed and 3-manifold – there are all topological properties, we discussed them above, feel free turn back. So all the Conjecture says is that if you have an object, and it meets these three properties, it is topologically the same as a 3-sphere, our stitched together globes.

What is the shape of the Universe?

Before we talk about the universe, let's talk about the Earth. Famously, there was a time when people thought the Earth had a boundary, an "edge" that you could fall off. It doesn't, but they also weren't crazy. It certainly looked flat. They also guessed that the planet had a finite surface area. Most things with finite surface areas have an edge, like the top of a table.

Then there was a period of time where we knew there was no boundary, but we still couldn't be sure of the planet's shape. Before spaceflight, how could we know for sure that we lived on a sphere rather than an enormous donut – so enormous that you couldn't see the other side? Well, from the above, we know one theoretical way to tell a sphere from a donut. We put our theoretical rubber band around every possible loop. If it always contracts to a point, then our world is simply connected and therefore a 2-sphere. Of course, this involves an infinite number of "rubber band checks" but I'm voluntarily writing a paper on topology, so clearly some people have spare time.

Perhaps it is useful to define what "The Universe" is. My favorite definition is that the Universe is everything that we can possibly, in theory, interact with. Faraway stars may be unreachable no matter the technology, but we can see their light and feel their gravitational. If it's another dimension and can't affect us – then that would be a different Universe.[8]

We know a lot about the universe. We can see light that originated just after it began, 13.8 billion years ago. We know the universe is expanding, so the stars that emitted this light are now about 47 billion light years away. That's an unimaginably huge distance. If you write it out in miles, the number has 24 digits. If you travelled fast enough to go from the Earth to the Sun and back *every second*,[9] it would still take you 47,000,000 years to get to the farthest points that we can see.

So it's huge – but what can we say about it topologically?

Let's start with the easy one: so far as we can tell, the universe is locally flat, or at least very close. There are a lot of experiments that tell us this, but I think the coolest is the BOOMERanG experiment.[10] The BOOMERanG sent a balloon telescope forty kilometers to measure huge triangles in space. Within experimental error, the angles totaled to 180°. The angles only total to 180° if the space is flat. Obviously, we tested only one point in the entire universe – but I'm pretty confident that the

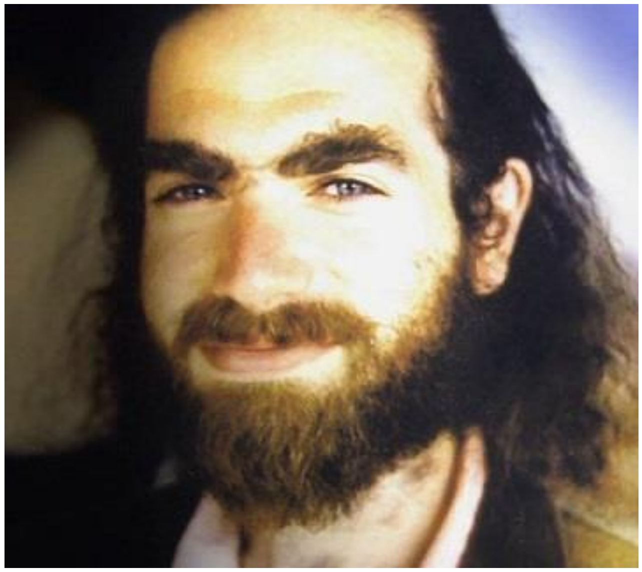
[8] There are a lot of good reasons to think that our Universe is one of many, part of the so-called Multi-verse. For more, see A Universe From Nothing, an excellent read.

[9] Which is impossible – it's much faster than the speed of light.

[10] More here, and of course follow the links on Wikipedia to really go down the rabbit hole.

local geometry of the universe is flat. Hence, the universe is a **3-manifold**.

Is the universe simply connected? Will the rubber band test always work? Well, our problem here is even worse than that of our ancestors – there is obviously no way to test every possible loop to ensure it collapses to a point.[11] But everything that we know – your day-to-day experiences, every observation we've taken – makes most people think the universe is **simply connected.**



GRIGORY PERELMAN

We don't know if the universe is finite or infinite; but speaking for myself I think it is finite.[12] Assume all matter began at a point – the basic concept of the Big Bang. You can draw a 3-sphere around this initial point; it will be finite in size. The universe started expanding. This expansion was very fast, but **finite**. Therefore, you can always draw a 3-sphere of finite size that includes all of the matter from the Big Bang. As long as the universe hasn't been around for an infinite amount of time, the final 3-sphere surrounding today's universe will still be finite. This argument is convincing on a superficial level, but no more, so no need to point out its many flaws.

An infinite universe has other problems – it would need to have an infinite amount of mass or energy or both. If the universe were truly infinite, that means that there are an infinite number of stars? If so, shouldn't the sky be brighter? This is Olbers's Paradox – but it can be solved by saying that the speed of light allows us to see only a finite subset of the infinite number of stars in existence. So in the end, speaking for myself, as a layman, I think it is more likely than not that the universe is **finite**.

When a person hears that the universe is expanding, they always ask the same question. If this is the case, then what is "outside" of the universe. But this is just a limitation of our ability to picture 3-dimensional spaces. The surface of our planet is a finite 2-dimensional shape; it doesn't have an "edge" and there is no way to get "outside" of it (in two dimensions). Similarly, the universe almost certainly has **no boundary**. Taking this with my guess of finite size means that the universe is likely **closed**.

Simply connected, closed, 3-manifold. We made some assumptions – wild ones perhaps – and mix in the Poincaré Conjecture. It seems likely that our universe is a simple 3-sphere.

So what? The universe is a 3-sphere? Why should you care?

As a theoretical mathematician (or somebody who once studied the topic), you are always asked "what can you do with it." The purpose of mathematics is to extend the boundaries of what we know, usually indifferent to the practical value. It is easy to find many of history's great mathematicians saying that they don't think about applications of their work. But more often than not, when you look at the

[11] We're talking about spacetime, so it's not a problem if our proverbial rubber band gets caught up by a donut, a planet, or even a donut-shaped planet.

[12] For all this talk about the shape of the universe, see A Universe From Nothing, above.

world 100 years later, you'll find their topic critical to everyday life.

The topological shape of the planet was clearly important to early explorers, even if the idea of "an edge" had disappeared long before Columbus. For the shape of the universe to matter, you'll have to wait a while. If you look out a few years – a decillion[13] or so – the universe will almost certainly reach an end state. What this end state is depends on truly weird things, like "phantom dark energy," as well as comparatively trivial ideas like the shape of the universe. Depending on the exact value of a few properties, we could be headed toward: a Big Freeze, where everything becomes part of black holes, which themselves disappear over time; a Big Rip, where the accelerating speed of expansion eventually tears all matter to sub-atomic particles; a Big Crunch, where gravity reverses expansion and pulls everything back to a single point, possibly leading to a new Big Bang; or the Big Slurp, which posits that space itself is unstable, just waiting for a singularity to "gulp" up the universe, destroying all the protons.

I'm not a cosmologist. I'm not even a topologist.[14] This isn't a peer-reviewed journal or a serious work of scholarship. Nothing in this Volume is even close to a proof that the universe is a 3-sphere. It's just my musings about the Poincaré Conjecture.

But if we could prove the universe's shape, it would tell us a lot about where we will end up. If the universe is a 3-sphere, the Big Crunch is off the table – it's already considered unlikely. If you are choosing between the Big Freeze and Big Rip, a 3-sphere universe makes the former more likely.[15] To vastly oversimplify - this is because a 3-sphere universe can be expected to continue expanding with accelerating too much.

Far too many people fear mathematics. Terms like topological space, homeomorphism and 3-manifold aren't exactly relatable. Maybe when you started this Volume, you were concerned, not sure you wanted to tackle it. It is not a simple topic. You should be happy with yourself trying something new and for making it to the end. Even if you didn't understand every word.

[13] That number has 33 zeros in it. In fact, that's on the earlier end of possible time-scales.

[14] Intro to Topology was a required class to complete my Course so many years ago. I signed up for it first term senior year; dropped it mid-way through because I was in danger of failing. Signed up again second term and battled my way to a B-minus.

[15] The Big Slurp is a new one that's coming from our discovery/understanding of the Higgs Boson. It is by far the weirdest and involves the entire universe disintegrating. This would start from a single "vacuum metastability event" and propagate out at the speed of light.